MERIA Scenario "Bicycle factory"

Linear and piecewise linear functions

Target knowledge	The construction of piecewise linear functions as an optimal solution to a problem with multiple linear conditions.
Broader goals	Drawing graphs of (linear) functions on paper and using ICT. Discussion about scaling the graphs along one axis. Deeper understanding of linear functions (the slope a and the constant b) by using them on linear conditions to construct piecewise linear functions. Discussion of the continuous and discrete aspects in relation to algebraic and graphical representations in the modelling process. Inquiry skills: experimenting with numbers before drawing graphs, disregarding unimportant data and obvious suboptimal factories, interpreting the results obtained in the modelling process, taking responsibility for the final report and presenting findings in a form of advice. Interdisciplinary skills: students may discuss about various economical aspects of the problem such as the difference between profit/earnings and revenue. Professional communication skills are emphasized in writing the report.
Prerequisite mathematical knowledge	Drawing the graph of a linear function. Familiarity with the notation $f(x) = ax + b$ and the interpretation of a and b.
Grade	Age 15 – 16, grade 9 – 10 (even earlier with smaller numbers)
Time	50 min (80 min)
Required material	The table with data about costs (see table below) Grid paper and/or applets (for changing the linear conditions) and/or ICT in general, for plotting, changing and adding conditions, finding intersections etc. A wide black or white board (or smartboard).

The table with data about costs:

Areas of location	Costs of building the factory in that area in €	The costs of producing one bicycle at the factory in €
A	300 000	120
B	450 000	110
C	660 000	60
D	680 000	80

Observations from implementation
The context of observations (grade, institution, country, etc.):

Problem:

You are a consultant who advises companies on where to run factory buildings for the production of bicycles. Based on the table showing the costs in different areas, what would you advise the companies and why?[1]

Phase	Teacher's actions incl. instructions	Students' actions and reactions	Observations from implementation
Devolution (didactical) 5 minutes	The teacher explains the situation and table above and poses the problem. "How would you in general guide the companies to place their factory? You should work with your neighbour and prepare to present your solution at the board later on."	Students listen, understand the relevance of the problem and feel engaged to work on it. They may have questions as to the meaning of the table and the problem. The teacher should explicitly give the students a chance to ask such questions, to make sure everyone understand the task.	
Action (adidactical) 15 (20) minutes	The teacher observes and notes how students approach the problem. Here the teacher gains knowledge about the students' prerequisite knowledge. It is important that the teacher does not give "hints" to the pairs, and avoids interaction with them except, if needed, to repeat the assignment.	The pairs start trying out different strategies or ideas based on their prerequisite knowledge. See "Possible ways for students to realize target knowledge" below. Because the students work in pairs, the adidactic formulation will occur.	

[1] The problem is inspired by Example 2.10 discussed in the book *„Primijenjena matematika podržana računalom"*, designed also by the author of this scenario in the scope of the project *„STEM genijalci*".

Formulation (didactical) 10 (15) minutes	The teacher chooses groups (at least 5) to present different strategies at the black/white board – the board should be divided into areas before the presentations. The students are not allowed to erase afterwards. Then, let the chosen pairs present orally, starting with simpler solutions. At this point, no validation is sought for.	Pairs are presenting accordingly to the teacher's plan (first, simple solutions based on numbers, then solutions with graphs and functions).	
Devolution (didactical) 1 minute	Discuss with your partner what similarities or differences you see in the presented work. Use this to improve your own answer to the management of the factory. I will ask you to report back after 5 (10) minutes.	Students listen. Teacher should make sure the students understand.	
Action / formulation (adidactical) 5 (15) minutes	The teacher circulates in the classroom to observe what the pairs had noticed and discussed, and how they make use of others' ideas.	The pairs are pointing to similarities and differences, trying to improve their own solution.	
Formulation and validation (didactical) 10 (15) minutes	The teacher calls on different pairs, to get as many observations and improved answers as possible. The teacher strives to have students identify any mistakes in previous solutions.	Students formulate similarities and differences and explain how they have improved their own solution by taking into account the work of the others; they may also point to shortcomings in some of this work.	
Institutionalisation (didactical) 5 (10) minutes	The teacher emphasises that there is not one correct answer, but the solution depends on how many bicycles are produced. The teacher first bases explanations on students' solutions on the board, then introduces the notation of	Students listen and recognize their own strategy in relation to the definition, and reflect on how this compares with the others. Students write their notes.	

	functions defined piecewise, using the example : $$f(x) = \begin{cases} 120x + 3 \cdot 10^5, x \leq a \\ 60x + 6.6 \cdot 10^5, x \geq a \end{cases}$$ where a=6000. (S)He uses this to summarize how to advise the company: area B and D are never optimal, while A and C are optimal for the production below and above 6000 bicycles, respectively. The optimal cost function is a piecewise linear function (defined on positive integers).		

Possible ways for students to realize target knowledge	• Some students begin working with some of the numbers, to see what they mean, such as: ○ Some students begin by calculating the price for concrete numbers of bicycles in each area. They may use trial and error to find numbers for which two areas give the same. ○ Students can create tables for each geographical area calculating the total costs for each number of bicycles comparing and point to the cheapest solution for any given number of bicycles (this can be done with pen and paper or in a spreadsheet environment). ○ Considering two locations, to cover the difference between the fixed costs with the difference between variable costs (e.g., to answer: how many bicycles must be produced before B is better than A); a total of six such comparisons are needed to provide a complete answer. • Some students take the function approach right away, and write down four equations, where each function represent the costs of the production of *x* bicycles: $f(x) = 120x + 300\,000,$ $g(x) = 110x + 450\,000,$ $h(x) = 60x + 660\,000,$ $k(x) = 80x + 680\,000.$ ○ The graphs of the functions are drawn in one or more coordinate system, and from the graphic representation students argue for the placement of the factory. ○ Students who use grid paper might read the point of intersection on coordinate axes.

- Students who use ICT may plot all the linear functions right away but will struggle to adjust axis to “see” them.
- In any case, the interpretation of the functions and the need to minimize the costs do not fall out automatically from any of the above but require thinking about the problem. Mistakes will occur, such as confusing production costs with the selling price or profit, etc.
- Based on the functional equations, the intersections between functions are found by equating pairs of equations. Students will make use of the graphic representation to know which pairs of equations are relevant. This strategy also requires equation-solving techniques.

- The students may reach different conclusions.
  - Whether the students work with numbers (and tables) or functions (and graphs) some will realize that there is not one “best area”, but the advice to be given depends on how many bicycles are produced. The conclusion could be more or less precise, formulated in words, equations, graphs, etc.
  - Some students will provide a quick and erroneous answer, e.g. “A is best because when we compute the cost to make 1, 2, …, 10 bicycles, we always get the cheapest price there.”

- **Example of graphs and equations which students could produce** (either on paper or, as here, using technology), to be used to identify how different areas are more economical at different values for the number of bicycles produced.